

TOWARD A THEORY OF NONPARAMETRIC MULTIPLE COMPARISONS

by

E. James Harner

BU-399-M

November, 1971

Abstract

Several procedures exist for making nonparametric multiple comparisons. None, however, have independent test statistics. The class of procedures in this paper is a generalization of the Mann-Whitney two-sample statistic. Under the null hypothesis these statistics are often independent.

The properties of a statistic which tests the ordering among three groups of populations is the main concern here. Recursion relationships enable the calculation of the small sample distribution. From this recursion certain moments and asymptotic normality of the statistic are derived.

TOWARD A THEORY OF NONPARAMETRIC MULTIPLE COMPARISONS

by

E. James Harner

BU-399-M

November, 1971

1. Introduction

The purpose of this paper is to begin a theory of nonparametric orthogonal comparisons. The procedure develops out of a consideration of tests of homogeneity against ordered alternatives [3], [4], i.e.,

$$H_0: F_1(x) = \dots = F_k(x) \quad \text{for all real } x.$$

$$H_a: F_1(x) \geq \dots \geq F_k(x) \quad \text{with at least one strict inequality.}$$

Previous approaches to this problem essentially rely on adding the $\binom{k}{2}$ two-sample Mann-Whitney statistics [1],

$$(1.1) \quad J = \sum_{i=1}^{k-1} \sum_{j=i+1}^k M_{ij}.$$

Jonckheere [3] gives an example which illustrates this hypothesis: Does degree of stress affect the performance of some task of manual dexterity? An experimental situation is counting the number of mistakes for increasing levels of stress.

This type of alternative is often meaningful, but in other cases prohibitively restrictive. Using (1.1) as a test statistic has several drawbacks also.

- 1) The values of J in the rejection region can be inconsistent with the alternative.
- 2) J may not detect alternatives with only one or a few strict inequalities.

As an alternative approach a procedure still in formative stages is a sequence of hypotheses. The alternatives will be one of two types:

$$(1.2) \quad (F_1, \dots, F_{k_1}) < \dots < (F_{k_1 + \dots + k_{h-1} + 1}, \dots, F_{k_1 + \dots + k_h})$$

$$(1.3) \quad [(F_1, \dots, F_{l_1}) < (F_{l_1 + 1}, \dots, F_{l_1 + l_2}) < (F_{l_1 + l_2 + 1}, \dots, F_{l_1 + l_2 + l_3})]$$

$$\cup [(F_1, \dots, F_{l_1}) > (F_{l_1 + 1}, \dots, F_{l_1 + l_2}) > (F_{l_1 + l_2 + 1}, \dots, F_{l_1 + l_2 + l_3})]$$

The second general alternative (1.3) means that $(F_{l_1 + 1}, \dots, F_{l_1 + l_2})$ are between (F_1, \dots, F_{l_1}) and $(F_{l_1 + l_2 + 1}, \dots, F_{l_1 + l_2 + l_3})$. If the alternative is true the group of populations within a set of parentheses is not necessarily homogeneous, additional hypotheses being necessary to determine their equality.

For the populations present in (1.2) let $n_1, \dots, n_{k_1}, \dots, n_{k_1 + \dots + k_h}$ be the sample sizes. The sample sizes of populations from (1.3) are $n_1, \dots, n_{l_1 + l_2 + l_3}$. In this paper the F_i 's are assumed absolutely continuous to avoid the problems of ties. The statistics that follow are definable by either ranks or the actual observations.

$$(1.4) \quad U_{(1, \dots, k_1) < \dots < (k_1 + \dots + k_{h-1} + 1, \dots, k_1 + \dots + k_h)} =$$

of h-tuples of obs'ns. such that the i^{th} obs'n.
belongs to the i^{th} group of pop'ns. and is the
 i^{th} smallest in the h-tuple.

$$(1.5) \quad U_{[(1, \dots, l_1) < (l_1 + 1, \dots, l_1 + l_2) < (l_1 + l_2 + 1, \dots, l_1 + l_2 + l_3)] \cup [(1, \dots, l_1) > (l_1 + 1, \dots, l_1 + l_2) > (l_1 + l_2 + 1, \dots, l_1 + l_2 + l_3)]} =$$

of obs'ns. from $(F_{l_1+1}, \dots, F_{l_1+l_2})$ which are between
obs'ns. from (F_1, \dots, F_{l_1}) and $(F_{l_1+l_2+1}, \dots, F_{l_1+l_2+l_3})$.

Several examples will illustrate the hypotheses and statistics.

Example 1.

$$H_1 = 2 = 3: \quad F_1 = F_2 = F_3$$

$$H_1 < (2, 3): \quad F_1 < (F_2, F_3).$$

The test for this hypothesis is the Mann-Whitney two-sample statistic, $U_1 < (2, 3)$, with sample sizes n_1 and $n_2 + n_3$. Under $H_1 < (2, 3)$ this statistic says nothing about the relative positions of F_2 and F_3 . If n_1 and $n_2 + n_3 \rightarrow \infty$ in any arbitrary way $U_1 < (2, 3)$ is asymptotically normal.

If $H_1 < (2, 3)$ is accepted then the following hypothesis is of interest:

$$H_2 = 3: \quad F_2 = F_3$$

$$H_2 < 3: \quad F_2 < F_3.$$

The test statistic is again a Mann-Whitney statistic, $U_2 < 3$. The two test statistics $U_1 < (2, 3)$ and $U_2 < 3$ are independent under the null hypothesis $F_1 = F_2 = F_3$ since a triplet of observations satisfying $1 < (2, 3)$ is (by Wald's theorem on order statistics) equally likely to satisfy $2 < 3$ or $3 < 2$. If $n_1 = n_2 = n_3 = 2$, for example, then the joint frequency distribution of $U_1 < (2, 3)$ and $U_2 < 3$ over the $(n_1 + n_2 + n_3)! / n_1! n_2! n_3! = 90$ possible outcomes is given by:

$$U_1 < (2, 3)$$

	0	1	2	3	4	5	6	7	8
0	1	1	2	2	3	2	2	1	1
1	1	1	2	2	3	2	2	1	1
2	2	2	4	4	6	4	4	2	2
3	1	1	2	2	3	2	2	1	1
4	1	1	2	2	3	2	2	1	1

$$U_2 < 3$$

This example illustrates a simplest type of nonparametric orthogonal comparison. Note that these hypotheses potentially give much more information than an alternative of completely ordered populations. Also the second criticism of (1.1) is overcome since the possibility exists of specifying alternatives of partial orderings.

Example 2.

$$H_1 = 2 = 3 = 4 : F_1 = F_2 = F_3 = F_4$$

$$H_1 < (2, 3) < 4 : F_1 < (F_2, F_3) < F_4$$

An alternative to the J statistic (1.1) is $U_1 < (2, 3) < 4$ with sample sizes $n_1, n_2 + n_3, n_4$. The main part of this paper develops this statistic. As will be seen $U_1 < (2, 3) < 4$ approaches a normal distribution as $n_1, n_2 + n_3, n_4 \rightarrow \infty$. However, subject to constraints on the relative sizes of $n_1, n_2 + n_3, n_4$.

If $H_1 < (2, 3) < 4$ is accepted, then the following hypothesis is of importance:

$$H_2 = 3 : F_2 = F_3$$

$$H_2 < 3 : F_2 < F_3$$

As before $U_2 < 3$ is independent of $U_1 < (2, 3) < 4$ and asymptotically normal.

The preceding two examples utilize statistics of the form (1.4). Example 3 has a statistic of the form (1.5).

Example 3.

$$H_1 = 2 = 3 : F_1 = F_2 = F_3$$

$$H[1 < 2 < 3] \cup [1 > 2 > 3] : [F_1 < F_2 < F_3] \cup [F_1 > F_2 > F_3]$$

The statistic $U[1 < 2 < 3] \cup [1 > 2 > 3]$ is probably asymptotically normal as seen in Section 2. The second hypothesis,

$$H_1 = 3 : F_1 = F_3$$

$$H_1 < 3 : F_1 < F_3 ,$$

has a different nested structure from the preceding examples and $U_1 < 3$ is no longer independent of $U[1 < 2 < 3] \cup [1 > 2 > 3]$. However, asymptotic independence appears likely.

These examples illustrate various multiple comparison possibilities. All alternatives are of the form (1.2) or (1.3). The independence of statistics in Examples 1 and 2 are just cases of a more general theorem. The theorem is difficult to express because of the notation.

Let $R\{ \quad \}$ express a relation among the elements within brackets. For example $R\{(1,2),3\}$ would mean some relation between (1,2) and 3.

Theorem 1. $U_R\{(1, \dots, k_1), (k_1+1, \dots, k_1+k_2), \dots, (k_1+\dots+k_{h-1}+1, \dots, k_1+\dots+k_h)\}$

is independent of $U_R\{1, \dots, k_1\}, U_R\{k_1+1, \dots, k_1+k_2\}, \dots,$

$U_R\{k_1+\dots+k_{h-1}+1, \dots, k_1+\dots+k_h\}$. Furthermore, the last h statistics are independent of each other. The proof will not be given here but uses the same type of

reasoning as Example 1. Note that the statistics in Example 3 are different from those in the theorem.

The remainder of this paper concerns itself with the statistic $U_1 < 2 < 3$. No work has been done for the cases with more than two inequalities, e.g., $U_1 < 2 < 3 < 4$. In these latter cases the statistic may suffer criticism 2) of the J statistic.

2. Difference equations for the joint frequency of

$$U_1 < 2 \text{ and } U_1 < 2 < 3$$

The case of primary concern in this paper is that of an ordered alternative among the three populations, i.e.,

$$\begin{aligned} H_1 = 2 = 3 : & \quad F_1 = F_2 = F_3 \\ H_1 < 2 < 3 : & \quad F_1 < F_2 < F_3 . \end{aligned}$$

In this case $X(i, j_i)$, $j_i = 1, \dots, n_i$ represent independent and identically distributed random variables from F_i , $i = 1, 2, 3$. Define

$$U_1 < 2 = \# (X(1, j_1) < X(2, j_2))$$

and

$$U_1 < 2 < 3 = \# (X(1, j_1) < X(2, j_2) < X(3, j_3)) .$$

We now examine an ordered sequence of observed $X(1, j_1)$'s, $X(2, j_2)$'s, and $X(3, j_3)$'s to develop the difference equations. Let $N_{n_1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3)$ represent the number of sequences in which $X(1, j_1) < X(2, j_2)$ occurs $U_1 < 2$ times

and simultaneously $X(1, j_1) < X(2, j_2) < X(3, j_3)$ occurs $U_1 < 2 < 3$ times. Since only the relations between $X(1, j_1)$, $X(2, j_2)$, and $X(3, j_3)$ matter, we can replace the $X(1, j_1)$'s by 1, the $X(2, j_2)$'s by 2, and the $X(3, j_3)$'s by 3. The ordered sequence of outcomes now has elements with values 1, 2, and 3.

Theorem 2. $N_{n_1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3)$ satisfies the difference equation

$$(2.1) \quad N_{n_1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3) = N_{n_1-1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3) \\ + N_{n_1, n_2-1, n_3} (U_1 < 2 - n_1, U_1 < 2 < 3) + N_{n_1, n_2, n_3-1} (U_1 < 2, U_1 < 2 < 3 \\ - U_1 < 2)$$

with initial conditions

$$N_{n_1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3) = 0 \text{ if } U_1 < 2 < 0 \text{ or } U_1 < 2 < 3 < 0$$

$$N_{0, n_2, n_3} (U_1 < 2, U_1 < 2 < 3) = 0 \text{ if } U_1 < 2 > 0 \text{ or } U_1 < 2 < 3 > 0$$

$$= (n_2 + n_3)! / n_2! n_3! \text{ if } U_1 < 2$$

$$= U_1 < 2 < 3 = 0$$

$$N_{n_1, 0, n_3} (U_1 < 2, U_1 < 2 < 3) = 0 \text{ if } U_1 < 2 > 0 \text{ or } U_1 < 2 < 3 > 0$$

$$= (n_1 + n_3)! / n_1! n_3! \text{ if } U_1 < 2$$

$$= U_1 < 2 < 3 = 0$$

$$N_{n_1, n_2, 0} (U_1 < 2, U_1 < 2 < 3) = 0 \text{ if } U_1 < 2 < 3 > 0$$

$$= N_{n_1-1, n_2, 0} (U_1 < 2, U_1 < 2 < 3)$$

$$+ N_{n_1, n_2-1, 0} (U_1 < 2 - n_1, U_1 < 2 < 3)$$

$$\text{if } U_1 < 2 < 3 = 0.$$

Proof. Let s_{n_1, n_2, n_3} be a realization of an ordered sequence. Then define

$$s_{n_1, n_2, n_3} (U_1 < 2, U_1 < 2 < 3) = \left\{ s_{n_1, n_2, n_3} \mid U_1 < 2^{(s_{n_1, n_2, n_3})} = U_1 < 2, \right. \\ \left. U_1 < 2 < 3^{(s_{n_1, n_2, n_3})} = U_1 < 2 < 3 \right\}$$

$$s_{n_1, n_2, n_3}^{(i)} (U_1 < 2, U_1 < 2 < 3) = \left\{ s_{n_1, n_2, n_3} \mid U_1 < 2^{(s_{n_1, n_2, n_3})} = U_1 < 2, \right. \\ \left. U_1 < 2 < 3^{(s_{n_1, n_2, n_3})} = U_1 < 2 < 3, \right.$$

and the last element of

$$s_{n_1, n_2, n_3} \text{ is an } i \}, i = 1, 2, 3.$$

$$D_{s_{n_1, n_2, n_3}}^{(1)} (U_1 < 2, U_1 < 2 < 3) = \left\{ s_{n_1-1, n_2, n_3} \mid \text{there exists } s_{n_1, n_2, n_3} \in \right.$$

$$s_{n_1, n_2, n_3}^{(1)} (U_1 < 2, U_1 < 2 < 3) \text{ with its first } n_1 + n_2 + n_3 - 1$$

elements the same as the $n_1 + n_2 + n_3 - 1$ elements of s_{n_1-1, n_2, n_3} , so

$$\text{that } U_1 < 2^{(s_{n_1-1, n_2, n_3})} = U_1 < 2 \text{ and } U_1 < 2 < 3^{(s_{n_1-1, n_2, n_3})} \\ = U_1 < 2 < 3 \}$$

$$D_{s_{n_1, n_2, n_3}}^{(2)} (U_1 < 2^{-n_1}, U_1 < 2 < 3) = \left\{ s_{n_1, n_2-1, n_3} \mid \text{there exists } s_{n_1, n_2, n_3} \in \right.$$

$$s_{n_1, n_2, n_3}^{(2)} (U_1 < 2, U_1 < 2 < 3) \text{ with its first } n_1 + n_2 + n_3 - 1 \text{ elements}$$

the same as the $n_1 + n_2 + n_3 - 1$ elements of s_{n_1, n_2-1, n_3} , so that

$$U_1 < 2^{(s_{n_1, n_2-1, n_3})} = U_1 < 2^{-n_1} \text{ and } U_1 < 2 < 3^{(s_{n_1, n_2-1, n_3})} \\ = U_1 < 2 < 3 \}$$

$$D_S^{(3)}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2) = \{s_{n_1, n_2, n_3-1} \mid \text{there exists}$$

$$s_{n_1, n_2, n_3} \in$$

$$S_{n_1, n_2, n_3}^{(3)}(U_1 < 2, U_1 < 2 < 3) \text{ with its first } n_1 + n_2 + n_3 - 1 \text{ elements}$$

the same as the $n_1 + n_2 + n_3 - 1$ elements of s_{n_1, n_2, n_3-1} , so that

$$\begin{aligned} U_1 < 2^{(s_{n_1, n_2, n_3-1})} &= U_1 < 2 \text{ and } U_1 < 2 < 3^{(s_{n_1, n_2, n_3-1})} \\ &= U_1 < 2 < 3 - U_1 < 2 \end{aligned}$$

For the first term on the right of (2.1) we need to show that

$$D_S^{(1)}(U_1 < 2, U_1 < 2 < 3) = S_{n_1-1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3) \text{ for all values of}$$

$$U_1 < 2 \text{ and } U_1 < 2 < 3. \quad s_{n_1-1, n_2, n_3} \in D_S^{(1)}(U_1 < 2, U_1 < 2 < 3)$$

$$\Rightarrow s_{n_1-1, n_2, n_3} \in S_{n_1-1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3) \text{ by definition. If } s'_{n_1-1, n_2, n_3}$$

$$\in S_{n_1-1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3) \text{ then by adding a 1 the resulting element is a}$$

$$\text{member of } S_{n_1, n_2, n_3}^{(1)}(U_1 < 2, U_1 < 2 < 3); \text{ hence, } s'_{n_1-1, n_2, n_3}$$

$$\in D_S^{(1)}(U_1 < 2, U_1 < 2 < 3).$$

Now consider the second term on the right of (2.1). Here we must show

$$D_S^{(2)}(U_1 < 2 - n_1, U_1 < 2 < 3) = S_{n_1, n_2-1, n_3}(U_1 < 2 - n_1, U_1 < 2 < 3). \text{ As}$$

$$\text{before } s_{n_1, n_2-1, n_3} \in D_S^{(2)}(U_1 < 2 - n_1, U_1 < 2 < 3) \Rightarrow s_{n_1, n_2-1, n_3}$$

$$\in S_{n_1, n_2-1, n_3}(U_1 < 2 - n_1, U_1 < 2 < 3). \text{ If } s'_{n_1, n_2-1, n_3} \in S_{n_1, n_2-1, n_3}$$

$$(U_1 < 2 - n_1, U_1 < 2 < 3) \text{ then by adding a 2 the resulting element is a member of}$$

$$S_{n_1, n_2-1, n_3}^{(2)}(U_1 < 2, U_1 < 2 < 3) \text{ and then } s'_{n_1, n_2-1, n_3} \in D_S^{(2)}(U_1 < 2 - n_1,$$

$$U_1 < 2 < 3).$$

Finally for the third term on the right of (2.1) we must show

$$D_{S_{n_1, n_2, n_3}}^{(3)}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2) = S_{n_1, n_2, n_3-1}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2).$$

If $s_{n_1, n_2, n_3} \in D_{S_{n_1, n_2, n_3}}^{(3)}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2)$ then $s_{n_1, n_2, n_3-1} \in S_{n_1, n_2, n_3-1}(U_1 < 2, U_1 < 2 < 3)$ by definition. If $s'_{n_1, n_2, n_3-1}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2)$ then by adding a 3 the resulting element is a member of $S_{n_1, n_2, n_3}^{(3)}(U_1 < 2, U_1 < 2 < 3)$ and then $s'_{n_1, n_2, n_3-1} \in D_{S_{n_1, n_2, n_3}}^{(3)}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2)$.

The initial conditions follow directly from definitions of $U_1 < 2$ and $U_1 < 2 < 3$.

Q.E.D.

We can now write down the corresponding probability difference equation. Let $P_{n_1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3)$ be the probability of a sequence in which $X(1, j_1) < X(2, j_2)$ occurs $U_1 < 2$ times and $X(1, j_1) < X(2, j_2) < X(3, j_3)$ occurs $U_1 < 2 < 3$ times. Each sequence of 1's, 2's and 3's of the $(n_1 + n_2 + n_3)!/n_1!n_2!n_3!$ sequence has equal probability under $H_1 = 2 = 3$. It follows that

$$\begin{aligned} (2.2) \quad P_{n_1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3) &= \frac{n_1}{n_1 + n_2 + n_3} P_{n_1-1, n_2, n_3}(U_1 < 2, U_1 < 2 < 3) \\ &+ \frac{n_2}{n_1 + n_2 + n_3} P_{n_1, n_2-1, n_3}(U_1 < 2, U_1 < 2 < 3) + \frac{n_3}{n_1 + n_2 + n_3} \\ &\times P_{n_1, n_2, n_3-1}(U_1 < 2, U_1 < 2 < 3 - U_1 < 2). \end{aligned}$$

From recursion equation (2.2) the small sample distribution of $U_1 < 2$ and in particular of $U_1 < 2 < 3$ is possible to calculate. The calculation of tables will be forthcoming in another report. The recurrence relation (2.2) also permits the derivation of moments and asymptotic normality of $U_1 < 2 < 3$. This will be done in the next two sections.

We now turn to the development of small sample theory for simple hypotheses of form (1.3). The simplest case is

$$H_1 = 2 = 3 : F_1 = F_2 = F_3$$

$$H_{[1 < 2 < 3] \cup [1 > 2 > 3]} : [F_1 < F_2 < F_3] \cup [F_1 > F_2 > F_3].$$

Note that the alternative says that F_2 is between F_1 and F_3 without regard to an ordering. If the alternative is true then we consider the hypothesis

$$H_1 = 3 : F_1 = F_3$$

$$H_1 < 3 : F_1 < F_3$$

Define in addition

$$U_{1 < 3} = \# (X(1, j_1) < X(3, j_3))$$

$$U_{3 < 2} = \# (X(3, j_3) < X(2, j_2))$$

and

$$\begin{aligned} U_{[1 < 2 < 3] \cup [1 > 2 > 3]} &= \# (X(1, j_1) < X(2, j_2) < X(3, j_3)) \\ &+ \# (X(1, j_1) > X(2, j_2) > X(3, j_3)) \end{aligned}$$

Recently Tom Beetle [5] communicated that $U_{1 < 3}$ and $U_{[1 < 2 < 3] \cup [1 > 2 > 3]}$ are both asymptotically normal by using a theorem of Hoeffding. $U_{1 < 3}$ certainly is by [1]. Presumably $U_{[1 < 2 < 3] \cup [1 > 2 > 3]}$ is since the first term of the definition is, as shown in a later section, and the second term is analogous. The

asymptotic independence of $U[1 < 2 < 3] U[1 > 2 > 3]$ under $H_1 = 2 = 3$ and of $U_1 < 3$ under both $H_1 = 3$ and $H_1 < 3$ is desirable for orthogonal multiple comparisons. Whether or not this is true has not been completely resolved. Using the recursion relation below we could derive the joint asymptotic distribution of $U_1 < 3$ and $U[1 < 2 < 3] U[1 > 2 > 3]$. Since they are probably jointly asymptotically normal the correlation would provide us the answer under $H_1 = 2 = 3$.

Using the same reasoning as in Theorem 2 we have

Theorem 3. $N_{n_1, n_2, n_3}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3], U_1 < 3)$ satisfies the difference equation

$$\begin{aligned} & N_{n_1, n_2, n_3}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3], U_1 < 3) \\ &= N_{n_1-1, n_2, n_3}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3] - U_3 < 2, U_1 < 3) \\ &+ N_{n_1, n_2-1, n_3}(U_1 < 2 - n_1, U_3 < 2 - n_3, U[1 < 2 < 3] U[1 > 2 > 3], U_1 < 3) \\ &+ N_{n_1, n_2, n_3-1}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3] - U_1 < 2, U_1 < 3 - n_1) \end{aligned}$$

with initial conditions similar to those in Theorem 1.

Proof. The argument is analogous to Theorem 1.

The corresponding probability recursion equation is

$$\begin{aligned} (2.3) \quad & P_{n_1, n_2, n_3}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3], U_1 < 3) \\ &= \frac{n_1}{n_1 + n_2 + n_3} P_{n_1-1, n_2, n_3}(U_1 < 2, U_3 < 2, U[1 < 2 < 3] U[1 > 2 > 3] - U_3 < 2, U_1 < 3) \end{aligned}$$

$$+ \frac{n_2}{n_1 + n_2 + n_3} P_{n_1, n_2 - 1, n_3} (U_1 < 2 - n_1, U_3 < 2 - n_3, U_{[1 < 2 < 3]} U_{[1 > 2 > 3]}, U_{1 < 3})$$

$$+ \frac{n_3}{n_1 + n_2 + n_3} P_{n_1, n_2, n_3 - 1} (U_1 < 2, U_3 < 2, U_{[1 < 2 < 3]} U_{[1 > 2 > 3]}^{-U_{1 < 2}, U_{1 < 3} - n_1}).$$

From (2.3) the small sample distribution of $U_{[1 < 2 < 3]} U_{[1 > 2 > 3]}$ and $U_{1 < 3}$ is easy to compute.

The results of this are easily extendible to more than three populations as mentioned in the introduction.

3. Derivation of joint moments for

$$U_1 < 2 \text{ and } U_1 < 2 < 3 \text{ under } H_1 = 2 = 3.$$

Since the probability of the j_1 th 1 preceding the j_2 th 2 is $1/2$ we obtain

$$E_{n_1, n_2, n_3} (U_1 < 2) = n_1 n_2 / 2.$$

Likewise since the probability of the j_1 th 1 preceding the j_2 th 2 preceding the j_3 th 3 is $1/6$ we obtain

$$E_{n_1, n_2, n_3} (U_{(1 < 2 < 3)}) = n_1 n_2 n_3 / 6.$$

These, of course, are under $H_1 = 2 = 3$.

The expressions for the joint central moments $E_{n_1, n_2, n_3} (u_1^i < 2, u_1^j < 2 < 3)$, $u_1 < 2 = U_1 < 2 - n_1 n_2 / 2$ and $u_1 < 2 < 3 = U_1 < 2 < 3 - n_1 n_2 n_3 / 6$, are obtainable from (2.2) in the form of another recurrence relationship.

$$\begin{aligned}
 (3.1) \quad & E_{n_1, n_2, n_3} (u_1^i < 2 u_1^j < 2 < 3) \\
 &= \frac{n_1}{n_1 + n_2 + n_3} \sum_{u_1 < 2, u_1 < 2 < 3} (u_1 < 2 - n_1 n_2 / 2)^i (u_1 < 2 < 3 - n_1 n_2 n_3 / 6)^j \\
 &\quad \times P_{n_1 - 1, n_2, n_3} (u_1 < 2, u_1 < 2 < 3) \\
 &+ \frac{n_2}{n_1 + n_2 + n_3} \sum_{u_1 < 2, u_1 < 2 < 3} (u_1 < 2 - n_1 n_2 / 2)^i (u_1 < 2 < 3 - n_1 n_2 n_3 / 6)^j \\
 &\quad \times P_{n_1, n_2 - 1, n_3} (u_1 < 2 - n_1, u_1 < 2 < 3) \\
 &+ \frac{n_3}{n_1 + n_2 + n_3} \sum_{u_1 < 2, u_1 < 2 < 3} (u_1 < 2 - n_1 n_2 / 2)^i (u_1 < 2 < 3 - n_1 n_2 n_3 / 6)^j \\
 &\quad \times P_{n_1, n_2, n_3 - 1} (u_1 < 2, u_1 < 2 < 3 - u_1 < 2) .
 \end{aligned}$$

We expand each term on the right-hand side of (3.1). The first term becomes

$$\begin{aligned}
 & \frac{n_1}{n_1 + n_2 + n_3} \sum_{u_1 < 2, u_1 < 2 < 3} (u_1 < 2 - (n_1 - 1)n_2 / 2 - n_2 / 2)^i (u_1 < 2 < 3 - (n_1 - 1)n_2 n_3 / 6 - n_2 n_3 / 6)^j \\
 &\quad \times P_{n_1 - 1, n_2, n_3} (u_1 < 2, u_1 < 2 < 3) \\
 &= \frac{n_1}{n_1 + n_2 + n_3} \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{(i+j) - (\alpha+\beta)} \left(\frac{n_2}{2}\right)^{i-\alpha} \left(\frac{n_2 n_3}{6}\right)^{j-\beta} \\
 &\quad \times E_{n_1 - 1, n_2, n_3} (u_1^\alpha < 2 u_1^\beta < 2 < 3) .
 \end{aligned}$$

The second term becomes

$$\begin{aligned} & \frac{n_2}{n_1+n_2+n_3} \sum_{U_1 < 2, U_1 < 2 < 3} \left(U_1 < 2 - \frac{n_1(n_2-1)}{2} + \frac{n_1}{2} \right)^i \left(U_1 < 2 < 3 - \frac{n_1(n_2-1)n_3}{6} - \frac{n_1n_3}{6} \right)^j \\ & \times P_{n_1, n_2-1, n_3} (U_1 < 2, U_1 < 2 < 3) \\ & = \frac{n_2}{n_1+n_2+n_3} \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{j-\beta} \left(\frac{n_1}{2} \right)^{i-\alpha} \left(\frac{n_1n_2}{6} \right)^{j-\beta} E_{n_1, n_2-1, n_3} (u_1^\alpha < 2, u_1^\beta < 2 < 3). \end{aligned}$$

The last term becomes

$$\begin{aligned} & \frac{n_3}{n_1+n_2+n_3} \sum_{U_1 < 2, U_1 < 2 < 3} \left(U_1 < 2 - \frac{n_1n_2}{2} \right)^i \left(U_1 < 2 < 3 - \frac{n_1n_2(n_3-1)}{6} + U_1 < 2 - \frac{n_1n_2}{2} \right. \\ & \quad \left. + \frac{n_1n_2}{3} \right)^j P_{n_1, n_2, n_3-1} (U_1 < 2, U_1 < 2 < 3) \\ & = \frac{n_3}{n_1+n_2+n_3} \sum_{U_1 < 2, U_1 < 2 < 3} \left(U_1 < 2 - \frac{n_1n_2}{2} \right)^i \sum_{\beta=0}^j \binom{j}{\beta} \left(U_1 < 2 < 3 - \frac{n_1n_2(n_3-1)}{6} \right)^{j-\beta} \left(U_1 < 2 \right. \\ & \quad \left. - \frac{n_1n_2}{2} + \frac{n_1n_2}{3} \right)^\beta P_{n_1, n_2, n_3-1} (U_1 < 2, U_1 < 2 < 3) \\ & = \frac{n_3}{n_1+n_2+n_3} \sum_{\beta=0}^j \sum_{\gamma=0}^\beta \binom{j}{\beta} \binom{\beta}{\gamma} \left(\frac{n_1n_2}{3} \right)^{\beta-\gamma} E_{n_1, n_2, n_3-1} (u_1^{i+\gamma} < 2, u_1^{j-\beta} < 2 < 3). \end{aligned}$$

The resulting recursion equation for $E_{n_1, n_2, n_3} (u_1^i < 2, u_1^j < 2 < 3)$ is

$$\begin{aligned} (3.2) \quad & E_{n_1, n_2, n_3} (u_1^i < 2, u_1^j < 2 < 3) \\ & = \frac{n_1}{n_1+n_2+n_3} \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{(i+j)-(\alpha+\beta)} \left(\frac{n_2}{2} \right)^{i-\alpha} \left(\frac{n_2n_3}{6} \right)^{j-\beta} E_{n_1-1, n_2, n_3} (u_1^\alpha < 2, u_1^\beta < 2 < 3) \end{aligned}$$

$$\begin{aligned}
 & + \frac{n_2}{n_1+n_2+n_3} \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{j-\beta} \left(\frac{n_1}{2}\right)^{i-\alpha} \left(\frac{n_1 n_3}{6}\right)^{j-\beta} E_{n_1, n_2-1, n_3} (u_1^\alpha < 2 \ u_1^\beta < 2 < 3) \\
 & + \frac{n_3}{n_1+n_2+n_3} \sum_{\beta=0}^j \sum_{\gamma=0}^{\beta} \binom{j}{\beta} \binom{\beta}{\gamma} \left(\frac{n_1 n_2}{3}\right)^{\beta-\gamma} E_{n_1, n_2, n_3-1} (u_1^{i+\gamma} < 2 \ u_1^{j-\beta} < 2 < 3)
 \end{aligned}$$

We now turn to the calculation of moments. From [2] we know $E_{n_1, n_2, n_3} (u_1^{i+1} < 2)$, $i = 2, 4$. By rearranging the terms in (3.2) such that all terms of degree $i + j$ in $u_1 < 2$ and $u_1 < 2 < 3$ are on the left, the resulting equation has known (previously computed) terms on the right. The method of solving is to substitute polynomials with unknown coefficients in the resulting equation.

The calculation of $E_{n_1, n_2, n_3} (u_1 < 2 \ u_1 < 2 < 3)$ will illustrate the technique.

$$\begin{aligned}
 (3.3) \quad & (n_1+n_2+n_3) E_{n_1, n_2, n_3} (u_1 < 2 \ u_1 < 2 < 3) - n_1 E_{n_1-1, n_2, n_3} (u_1 < 2 \ u_1 < 2 < 3) \\
 & - n_2 E_{n_1, n_2-1, n_3} (u_1 < 2 \ u_1 < 2 < 3) - n_3 E_{n_1, n_2, n_3-1} (u_1 < 2 \ u_1 < 2 < 3) \\
 & = n_1 \sum_{\alpha=0}^1 \sum_{\substack{\beta=0 \\ \alpha+\beta < 2}}^1 \binom{1}{\alpha} \binom{1}{\beta} (-1)^{2-(\alpha+\beta)} \left(\frac{n_2}{2}\right)^{1-\alpha} \left(\frac{n_2 n_3}{6}\right)^{1-\beta} E_{n_1-1, n_2, n_3} (u_1^\alpha < 2 \ u_1^\beta < 2 < 3) \\
 & + n_2 \sum_{\alpha=0}^1 \sum_{\substack{\beta=0 \\ \alpha+\beta < 2}}^1 \binom{1}{\alpha} \binom{1}{\beta} (-1)^{1-\beta} \left(\frac{n_1}{2}\right)^{1-\alpha} \left(\frac{n_1 n_3}{6}\right)^{1-\beta} E_{n_1, n_2-1, n_3} (u_1^\alpha < 2 \ u_1^\beta < 2 < 3) \\
 & + n_3 \sum_{\beta=1}^1 \sum_{\gamma=0}^{\beta} \binom{1}{\beta} \binom{\beta}{\gamma} \left(\frac{n_1 n_2}{3}\right)^{\beta-\gamma} E_{n_1, n_2, n_3-1} (u_1^{i+\gamma} < 2 \ u_1^{j-\beta} < 2 < 3) \\
 & = n_1 [n_2^2 n_3 / 12] - n_2 [n_1^2 n_3 / 12] + n_3 [n_1 n_2 (n_1 + n_2 + 1) / 12] \\
 & = n_1 n_2 n_3 [n_2 / 6 + 1 / 12] .
 \end{aligned}$$

Define $Q_{n_1 n_2 n_3} = n_2/6 + 1/12$ and

$$(3.4) \quad E_{n_1, n_2, n_3} \left(u_1 < 2 \quad u_1 < 2 < 3 \right) = n_1 n_2 n_3 \sum_{i, j, k} a_{ijk} n_1^i n_2^j n_3^k .$$

Substituting (3.4) in (3.3) we obtain

$$\begin{aligned} & (n_1 + n_2 + n_3) \sum a_{ijk} n_1^i n_2^j n_3^k - (n_1 - 1) \sum a_{ijk} (n_1 - 1)^i n_2^j n_3^k \\ & - (n_2 - 1) \sum a_{ijk} n_1^i (n_2 - 1)^j n_3^k - (n_3 - 1) \sum a_{ijk} n_1^i n_2^j (n_3 - 1)^k = Q_{n_1 n_2 n_3} . \end{aligned}$$

Expanding the left-hand side gives us

$$\begin{aligned} & \sum a_{ijk} \left[\sum_{\alpha=0}^i \binom{i+1}{\alpha} (-1)^{i-\alpha} n_1^{\alpha} n_2^j n_3^k + \sum_{\beta=0}^j \binom{j+1}{\beta} (-1)^{j-\beta} n_1^i n_2^{\beta} n_3^k + \sum_{\gamma=0}^k \binom{k+1}{\gamma} (-1)^{k-\gamma} n_1^i n_2^j n_3^{\gamma} \right] \\ & = Q_{n_1 n_2 n_3} . \end{aligned}$$

The solution is

$$a_{000} [1 + 1 + 1] + a_{010} [n_2 - 1 + 2n_2 + n_2] = n_2/6 + 1/12$$

$$a_{010} = 1/24$$

$$a_{000} = 1/24 .$$

Then

$$E_{n_1, n_2, n_3} \left(u_1 < 2 \quad u_1 < 2 < 3 \right) = n_1 n_2 n_3 (n_2 + 1)/24 .$$

Using this technique the moments for $i+j = 2, 3$ are

$$E_{n_1, n_2, n_3} \left(u_1^2 < 2 \right) = n_1 n_2 (n_1 + n_2 + 1)/12$$

$$\begin{aligned} E_{n_1, n_2, n_3} \left(u_1^2 < 2 < 3 \right) &= (1/180) n_1 n_2 n_3 [4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 \\ &\quad + 5n_3 + 4] \end{aligned}$$

$$E_{n_1, n_2, n_3} \left(u_1 < 2 \quad u_1 < 2 < 3 \right) = n_1 n_2 n_3 (n_2 + 1) / 24$$

$$E_{n_1, n_2, n_3} \left(u_1^2 < 2 \quad u_1 < 2 < 3 \right) = \left(1/1440 \right) n_1 n_2 n_3 \left[6 - 17n_1 + 7n_2 - 8n_1 n_2 \right. \\ \left. - 20n_1^2 - 8n_2^2 \right]$$

$$E_{n_1, n_2, n_3} \left(u_1 < 2 \quad u_1^2 < 2 < 3 \right) = \left(1/4320 \right) n_1 n_2 n_3 \left[-6 + 39n_1 + 18n_2 + 7n_3 \right. \\ \left. + 84n_1 n_2 + 30n_2 n_3 + 34n_1^2 + 12n_2^2 + 36n_1 n_2^2 + 24n_1^2 n_2 + 38n_1^2 n_3 \right]$$

$$E_{n_1, n_2, n_3} \left(u_1^3 < 2 < 3 \right) = n_1 n_2 n_3 \left[n_2^2 n_3^2 \left(1/1080 \right) + n_1^2 n_3^2 \left(53/15120 \right) + n_1^2 n_2^2 \left(2/945 \right) \right. \\ \left. + n_1^2 n_2 n_3 \left(1/252 \right) + n_1 n_2^2 n_3 \left(31/2520 \right) + n_1 n_2 n_3^2 \left(1/252 \right) + n_2^2 n_3^2 \left(13/2016 \right) \right. \\ \left. + n_2^2 n_3 \left(1/5040 \right) + n_1 n_2^2 \left(29/10,080 \right) + n_1 n_2^2 \left(1/1260 \right) + n_1^2 n_3 \left(9811/1,723,680 \right) \right. \\ \left. + n_1^2 n_2 \left(23/2520 \right) + n_1 n_2 n_3 \left(9/280 \right) + n_1^2 \left(277/25,200 \right) + n_2^2 \left(121/75,600 \right) \right. \\ \left. + n_3^2 \left(59/30,240 \right) + n_1 n_2 \left(11/2520 \right) + n_1 n_3 \left(37/7200 \right) + n_2 n_3 \left(1/5040 \right) \right. \\ \left. + n_1 \left(33/2800 \right) + n_2 \left(-41/25,200 \right) + n_3 \left(85,009/34,473,600 \right) \right. \\ \left. - \left(8,081,197/1,075,576,320 \right) \right]$$

$$E_{n_1, n_2, n_3} \left(u_1^3 < 2 \right) = 0 \text{ by symmetry.}$$

4. Asymptotic Normality

As in [2] we define an operator ψ on $F(n_1, n_2, n_3)$, a function of integers.

$$(4.1) \quad \psi F(n_1, n_2, n_3) \equiv n_1 \left[F(n_1, n_2, n_3) - F(n_1 - 1, n_2, n_3) \right] + n_2 \left[F(n_1, n_2, n_3) \right. \\ \left. - F(n_1, n_2 - 1, n_3) \right] + n_3 \left[F(n_1, n_2, n_3) - F(n_1, n_2, n_3 - 1) \right].$$

Equation (3.3) is a special case of a more general theory.

$$\begin{aligned}
 & \psi E_{n_1, n_2, n_3} (u_1^i < 2 \ u_1^j < 2 < 3) \\
 &= (n_1 + n_2 + n_3) E_{n_1, n_2, n_3} (u_1^i < 2 \ u_1^j < 2 < 3) - n_1 E_{n_1-1, n_2, n_3} (u_1^i < 2 \ u_1^j < 2 < 3) \\
 & \quad - n_2 E_{n_1, n_2-1, n_3} (u_1^i < 2 \ u_1^j < 2 < 3) - n_3 E_{n_1, n_2, n_3-1} (u_1^i < 2 \ u_1^j < 2 < 3) \\
 &= n_1 \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{(i+j)-(\alpha+\beta)} \left(\frac{n_2}{2}\right)^{i-\alpha} \left(\frac{n_2 n_3}{6}\right)^{j-\beta} E_{n_1-1, n_2, n_3} (u_1^\alpha < 2 \ u_1^\beta < 2 < 3) \\
 & \quad + n_2 \sum_{\alpha=0}^i \sum_{\beta=0}^j \binom{i}{\alpha} \binom{j}{\beta} (-1)^{j-\beta} \left(\frac{n_1}{2}\right)^{i-\alpha} \left(\frac{n_1 n_3}{6}\right)^{j-\beta} E_{n_1, n_2-1, n_3} (u_1^\alpha < 2 \ u_1^\beta < 2 < 3) \\
 & \quad + n_3 \sum_{\beta=1}^j \sum_{\nu=0}^{\beta} \binom{j}{\beta} \binom{\beta}{\nu} \left(\frac{n_1 n_2}{3}\right)^{\beta-\nu} E_{n_1, n_2, n_3-1} (u_1^{i+\nu} < 2 \ u_1^{j-\beta} < 2 < 3) .
 \end{aligned}$$

ψ has several nice properties which the following lemmas express.

Lemma 1. If $\psi F(n_1, n_2, n_3)$ is a polynomial of degree q in all variables, r in n_1 , s in n_2 , t in n_3 , then $F(n_1, n_2, n_3)$ is a polynomial of degree q in all variables, r in n_1 , s in n_2 and t in n_3 .

Proof. See [2].

Lemma 2. If V_q, W_q are polynomials of degree q and if $\psi V_q / W_q \rightarrow c$ as $n_1, n_2, n_3 \rightarrow \infty$ in such a way that $f(n_1, n_2, n_3) \rightarrow f_0$ then

$$\lim_{f(n_1, n_2, n_3) \rightarrow f_0} \frac{V_q}{W_q} = c/q$$

Proof. See [2].

These properties are necessary in the derivation of asymptotic normality.

The following result is partially a result of lemma 1.

Lemma 3. $E_{n_1, n_2, n_3} \left(u_1^i < 2 \ u_1^j < 2 < 3 \right)$ is of degree $(3/2)i + (5/2)j$ in all variables if $i + j$ is even, of degree $(3/2)i + (5/2)j - 1/2$ in all variables if $i + j$ is odd*, of degree $i + j$ in n_1 , of degree $i + j$ in n_2 , and of degree j in n_3 .

Proof. For $i + j = 2, 3$ the results of the lemma hold for $E_{n_1, n_2, n_3} \left(u_1^i < 2 \ u_1^j < 2 < 3 \right)$. Assume these conclusions to hold for $i + j \leq m$. Call this assumption I.

We need to consider both cases of $i + j$ odd and $i + j$ even.

Case I. $i + j = m$ even ($2n$ say).

$E_{n_1, n_2, n_3} \left(u_1^{2n} < 2 \right)$ is of degree $3n$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and 0 in n_3 by [1]. For $E_{n_1, n_2, n_3} \left(u_1^{2n-j} < 2 \ u_1^j < 2 < 3 \right)$ with $j < h$ we assume the results of the lemma. Call this assumption II.

Now write

$$(4.2) \quad \psi E_{n_1, n_2, n_3} \left(u_1^{2n-h} < 2 \ u_1^h < 2 < 3 \right) = A + B + C$$

where

$$\begin{aligned} A &= n_1 \left[(-1)^{2n} \left(\frac{n_2}{2} \right)^{2n-h} \left(\frac{n_2 n_3}{6} \right)^h \right. \\ &\quad + \dots + \frac{(2n-h)!}{(2n-h-k_1)!} \binom{h}{h-k_1} (-1)^{k_1+l_1} \left(\frac{n_2}{2} \right)^{k_1} \left(\frac{n_2 n_3}{6} \right)^{l_1} E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-k_1} < 2 \ u_1^{h-l_1} < 2 < 3 \right) \\ &\quad + \dots + \frac{(2n-h)(2n-h-1)}{2} (-1)^2 \left(\frac{n_2}{2} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-2} < 2 \ u_1^h < 2 < 3 \right) \\ &\quad + (2n-h)h (-1)^2 \left(\frac{n_2}{2} \right) \left(\frac{n_2 n_3}{6} \right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-1} < 2 \ u_1^{h-1} < 2 < 3 \right) \\ &\quad \left. + \frac{h(h-1)}{2} (-1)^2 \left(\frac{n_2 n_3}{6} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-h} < 2 \ u_1^{h-2} < 2 < 3 \right) \right] \end{aligned}$$

* $E_{n_1, n_2, n_3} \left(u_1^{i+j} < 2 \right) = 0$ for $i + j$ odd by symmetry.

$$\begin{aligned}
 & + (2n-h)(-1)\left(\frac{n_2}{2}\right)E_{n_1-1, n_2, n_3}\left(u_1^{2n-h-1} u_1^h \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + h(-1)\left(\frac{n_2 n_3}{6}\right)E_{n_1-1, n_2, n_3}\left(u_1^{2n-h} u_1^{h-1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 B = & n_2 \left[(-1)^h \left(\frac{n_1}{2}\right)^{2n-h} \left(\frac{n_1 n_3}{6}\right)^h \right. \\
 & + \dots + \binom{2n-h}{2n-h-k_2} \binom{h}{h-\ell_2} (-1)^{\ell_2} \left(\frac{n_1}{2}\right)^{k_2} \left(\frac{n_1 n_3}{6}\right)^{\ell_2} E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-k_2} u_1^{h-\ell_2} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + \dots + \frac{(2n-h)(2n-h-1)}{2} \left(\frac{n_1}{2}\right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-2} u_1^h \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + (2n-h)h(-1)\left(\frac{n_1}{2}\right)\left(\frac{n_1 n_3}{6}\right)E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-1} u_1^{h-1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + \frac{h(h-1)}{2} (-1)^2 \left(\frac{n_1 n_3}{6}\right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-h} u_1^{h-2} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + (2n-h)\left(\frac{n_1}{2}\right)E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-1} u_1^h \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + h(-1)\left(\frac{n_1 n_3}{6}\right)E_{n_1, n_2-1, n_3} \left(u_1^{2n-h} u_1^{h-1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \Big] \\
 C = & n_3 \left[h \left(\frac{n_1 n_2}{3}\right)E_{n_1, n_2, n_3-1} \left(u_1^{2n-h} u_1^{h-1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \right. \\
 & + hE_{n_1, n_2, n_3-1} \left(u_1^{2n-h+1} u_1^{h-1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + \frac{h(h-1)}{2} \left(\frac{n_1 n_2}{3}\right)^2 E_{n_1, n_2, n_3-1} \left(u_1^{2n-h} u_1^{h-2} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + \dots + \binom{h}{\ell_3} \binom{\ell_3}{k_3} \left(\frac{n_1 n_3}{3}\right)^{\ell_3-k_3} E_{n_1, n_2, n_3-1} \left(u_1^{2n-h+k_3} u_1^{h-\ell_3} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \\
 & + \dots + E_{n_1, n_2, n_3-1} \left(u_1^{2n} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right) \Big]
 \end{aligned}$$

In (4.2) we first examine A. If $k_1 + \ell_1 \geq 2$ and even then by assumption I

$E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-k_1} u_1^{h-\ell_1} \begin{smallmatrix} 1 & 2 & 3 \\ < & < & < \end{smallmatrix}\right)$ times its multiplying factor has degree

$(3/2)(2n-h) + (5/2)h - (1/2)k_1 - (1/2)l_1 + 1$ in all variables, $2n-k_1-l_1 + 1$ in n_1 , $2n$ in n_2 , and h in n_3 . Thus all terms with $k_1+l_1 > 2$ and even have degree less than $(3/2)(2n-h) + (5/2)h$ in all variables, less than $2n$ in n_1 , $2n$ in n_2 , and h in n_3 . Terms with $k_1+l_1 = 2$ have degree $(3/2)(2n-h) + (5/2)h$ in all variables, $2n-1$ in n_1 , $2n$ in n_2 , and h in n_3 . For $k_1+l_1 \geq 1$ and odd then by assumption I $E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-k_1} u_1^{h-l_1} \right)$ times its multiplying factor has degree $(3/2)(2n-h) + (5/2)h - (1/2)k_1 - (1/2)l_1 + (1/2)$ in all variables, $2n-k_1-l_1 + 1$ in n_1 , $2n$ in n_2 , and h in n_3 . All terms with $k_1+l_1 \geq 3$ and odd then have degree less than $(3/2)(2n-h) + (5/2)h$ in all variables, less than $2n$ in n_1 , $2n$ in n_2 , and h in n_3 . But terms with $k_1+l_1 = 1$ have degree $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and h in n_3 . However, the terms of highest degree will cancel with similar terms in B and C of (4.2).

We now consider B of (4.2). If $k_2+l_2 > 2$ then by assumption I $E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-k_2} u_1^{h-l_2} \right)$ times its multiplying factor has degree less than $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , less than $2n$ in n_2 , and h in n_3 . For $k_2+l_2 = 1, 2$ the terms are of degree $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 ($2n-1$ if $k_2+l_2 = 2$), and h in n_3 .

C of (4.2) is slightly different. For $k_3 = l_3 = k > 1$ by assumption II $E_{n_1, n_2, n_3-1} \left(u_1^{2n-h+k} u_1^{h-k} \right)$ times its multiplying factor has degree less than $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and less than h in n_3 . Likewise, the term with $k_3 = l_3 = 1$ have degree $(3/2)(2n-h) + (5/2)h$ in all variables $2n$ in n_1 , $2n$ in n_2 and h in n_3 . For $k_3 < l_3$ and $k_3 + l_3 \geq 3$ by assumption I $E_{n_1, n_2, n_3-1} \left(u_1^{2n-h+k_3} u_1^{h-l_3} \right)$ times its multiplying factor has degree less than $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and less than h in n_3 . Finally for $k_3 = 0, l_3 = 1$ and $k_3 = 0, l_3 = 2$ the corresponding terms have degree $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and h in n_3 ($h-1$ for $k_3 = 0, l_3 = 2$).

The terms of highest degree in A and B from (4.2) with $k_1=k_2=1$ and $\ell_1=\ell_2=0$ cancel. The sum of these terms is

$$\begin{aligned} & (2n-h)(-1)\left(\frac{n_1 n_2}{2}\right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-1} u_1^h \mid u_1 < 2 < 3 \right) \\ & + (2n-h)\left(\frac{n_1 n_2}{2}\right) E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-1} u_1^h \mid u_1 < 2 < 3 \right) \end{aligned}$$

The different subscripts on E do not affect the fact that the terms of highest degree in this sum cancel. Likewise the terms of highest degree with $k_1 = k_2 = k_3 = 0$ and $\ell_1 = \ell_2 = \ell_3 = 1$ from A, B, and C of (4.2) cancel.

$$\begin{aligned} & h(-1)\left(\frac{n_1 n_2 n_3}{6}\right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) \\ & + h(-1)\left(\frac{n_1 n_2 n_3}{6}\right) E_{n_1, n_2-1, n_3} \left(u_1^{2n-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) \\ & + h\left(\frac{n_1 n_2 n_3}{3}\right) E_{n_1, n_2, n_3-1} \left(u_1^{2n-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) . \end{aligned}$$

By lemma 1, therefore, $E_{n_1, n_2, n_3} \left(u_1^{2n-h} u_1^h \mid u_1 < 2 < 3 \right)$ has degree $(3/2)(2n-h) + (5/2)h$ in all variables, $2n$ in n_1 , $2n$ in n_2 , and h in n_3 .

Case II. $i+j = m$ odd ($2n+1$ say)

$E_{n_1, n_2, n_3} \left(u_1^{2n+1} \right) = 0$ by symmetry. We start the induction proof with $E_{n_1, n_2, n_3} \left(u_1^{2n} \right)$.

We write

$$(4.3) \quad \psi E_{n_1, n_2, n_3} \left(u_1^{2n+1-h} u_1^h \mid u_1 < 2 < 3 \right) = A' + B' + C'$$

where

$$\begin{aligned}
 A' &= n_1 \left[(-1)^{2n+1} \left(\frac{n_2}{2} \right)^{2n+1-h} \left(\frac{n_2 n_3}{6} \right)^h \right. \\
 &\quad + \dots + \binom{2n+1-h}{2n+1-h-k_1} \binom{h}{h-\ell_1} (-1)^{k_1+\ell_1} \left(\frac{n_2}{2} \right)^{k_1} \left(\frac{n_2 n_3}{6} \right)^{\ell_1} \\
 &\quad \times E_{n_1-1, n_2, n_3} \left(u_1^{2n+1-h-k_1} u_1^{h-\ell_1} \mid u_1 < 2 < 3 \right) \\
 &\quad + \dots + \frac{(2n+1-h)(2n-h)}{2} (-1)^2 \left(\frac{n_2}{2} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-h-1} u_1^h \mid u_1 < 2 < 3 \right) \\
 &\quad + (2n+1-h)h (-1)^2 \left(\frac{n_2}{2} \right) \left(\frac{n_2 n_3}{6} \right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) \\
 &\quad + \frac{h(h-1)}{2} (-1) \left(\frac{n_2 n_3}{6} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n+1-h} u_1^{h-2} \mid u_1 < 2 < 3 \right) \\
 &\quad + (2n+1-h) (-1)^2 \left(\frac{n_2}{2} \right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-h} u_1^h \mid u_1 < 2 < 3 \right) \\
 &\quad \left. + h (-1) \left(\frac{n_2 n_3}{6} \right) E_{n_1-1, n_2, n_3} \left(u_1^{2n+1-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) \right] \\
 B' &= n_2 \left[(-1)^h \left(\frac{n_1}{2} \right)^{2n+1-h} \left(\frac{n_1 n_3}{6} \right)^h \right. \\
 &\quad + \dots + \binom{2n+1-h}{2n+1-h-k_2} \binom{h}{h-\ell_2} (-1)^{\ell_2} \left(\frac{n_1}{2} \right)^{k_2} \left(\frac{n_1 n_3}{6} \right)^{\ell_2} E_{n_1, n_2-1, n_3} \left(u_1^{2n+1-h-k_2} u_1^{h-\ell_2} \mid u_1 < 2 < 3 \right) \\
 &\quad + \dots + \frac{(2n+1-h)(2n-h)}{2} \left(\frac{n_1}{2} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-h-1} u_1^h \mid u_1 < 2 < 3 \right) \\
 &\quad + (2n+1-h)h (-1) \left(\frac{n_1}{2} \right) \left(\frac{n_1 n_3}{6} \right) E_{n_1, n_2-1, n_3} \left(u_1^{2n-h} u_1^{h-1} \mid u_1 < 2 < 3 \right) \\
 &\quad + \frac{h(h-1)}{2} (-1)^2 \left(\frac{n_1 n_3}{6} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n+1-h} u_1^{h-2} \mid u_1 < 2 < 3 \right) \\
 &\quad \left. + \dots \right]
 \end{aligned}$$

$$\begin{aligned}
& + (2n+1-h) \binom{n_1}{2} E_{n_1, n_2-1, n_3} \left(u_1^{2n-h} u_1^h \right) \\
& + h(-1) \binom{n_1 n_3}{6} E_{n_1, n_2-1, n_3} \left(u_1^{2n+1-h} u_1^{h-1} \right) \Big] \\
C' = & n_3 \left[h \binom{n_1 n_2}{3} E_{n_1, n_2, n_3-1} \left(u_1^{2n+1-h} u_1^{h-1} \right) \right. \\
& + h E_{n_1, n_2, n_3-1} \left(u_1^{2n-h+2} u_1^{h-1} \right) \\
& + \frac{h(h-1)}{2} \left(\frac{n_1 n_2}{3} \right)^2 E_{n_1, n_2, n_3-1} \left(u_1^{2n+1-h} u_1^{h-2} \right) \\
& + \dots + \binom{h}{\ell_3} \binom{\ell_3}{k_3} \left(\frac{n_1 n_2}{3} \right)^{\ell_3-k_3} E_{n_1, n_2, n_3-1} \left(u_1^{2n+1-h+k_3} u_1^{h-\ell_3} \right) \\
& \left. + \dots + E_{n_1, n_2, n_3-1} \left(u_1^{2n+1} \right) \right] .
\end{aligned}$$

This induction starts with $h = 1$ in (4.3). The demonstration of this case is identical to the succeeding general h except that only assumption I is necessary. A' and B' of (4.3) are the same as the general h and thus are not given here. For $h = 1$,

$$\begin{aligned}
C' = & n_3 \left[\left(\frac{n_1 n_2}{3} \right) E_{n_1, n_2, n_3-1} \left(u_1^{2n} \right) \right. \\
& \left. + E_{n_1, n_2, n_3-1} \left(u_1^{2n+1} \right) \right] .
\end{aligned}$$

$E_{n_1, n_2, n_3-1} \left(u_1^{2n+1} \right) = 0$ by symmetry arguments. By assumption I $E_{n_1, n_2, n_3-1} \left(u_1^{2n} \right)$ times its multiplying factor has degree $3n + 3$ in all variables $2n + 1$ in n_1 , $2n + 1$ in n_2 , and 1 in n_3 . The highest order cancels, however, with corresponding terms

in A' and B' of (4.3) as in Case I. From the results of A', B', and C' of (4.3) we can state that $\psi_{E_{n_1, n_2, n_3}}^{2n} \left(u_1^{2n} \right)_{1 < 2 < 3}$ has degree $3n + 2$ in all variables, $2n + 1$ in n_1 , $2n + 1$ in n_2 , and 1 in n_3 and thus by lemma 1 $E_{n_1, n_2, n_3}^{2n} \left(u_1^{2n} \right)_{1 < 2 < 3}$ does.

With general h for $j < h$ assume the results of the lemma. Call this assumption III.

First consider A' of (4.3). If $k_1 + \ell_1 \geq 2$ and even then by assumption I $E_{n_1-1, n_2, n_3}^{2n+1-h-k_1} \left(u_1^{2n+1-h-k_1} u_1^{h-\ell_1} \right)_{1 < 2 < 3}$ times its multiplying factor has degree $3n+h - (1/2)k_1 - (1/2)\ell_1 + 2$ in all variables, $2n-k_1-\ell_1+2$ in n_1 , $2n+1$ in n_2 , and h in n_3 . The degree of terms with $k_1+\ell_1 = 2$ is $3n+h+1$ in all variables, $2n$ in n_1 , $2n+1$ in n_2 , and h in n_3 ; terms with $k_1+\ell_1 > 2$ have degree less than $3n-h+1$ in all variables, less than $2n+1$ in n_1 , $2n+1$ in n_2 , and h in n_3 . For $k_1+\ell_1 \geq 1$ and odd then by assumption I, $E_{n_1-1, n_2, n_3}^{2n+1-h-k_1} \left(u_1^{2n+1-h-k_1} u_1^{h-\ell_1} \right)_{1 < 2 < 3}$ times its multiplying factor has degree $3n+h - (1/2)k_1 - (1/2)\ell_1 + 5/2$ in all variables, $2n-k_1-\ell_1+2$ in n_1 , $2n+1$ in n_2 , and h in n_3 . The terms with $k_1+\ell_1 > 3$ and odd thus have degree less than $3n-h+1$ in all variables, less than $2n+1$ in n_1 , $2n+1$ in n_2 , h in n_3 . Terms with $k_1+\ell_1 = 3$ have degree $3n-h+1$ in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , and h in n_3 . For $k_1+\ell_1 = 1$, the terms have degree $3n-h+2$ in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , and h in n_3 . The terms of highest order with $k_1+\ell_1 = 1$ cancel with corresponding terms in B' and C' of (4.3).

The analysis for B' of (4.3) is exactly analogous to that of A' except with k_2 and ℓ_2 replacing k_1 and ℓ_1 . C' once again is somewhat different. For $k_3=\ell_3=k > 1$ by assumption III, $E_{n_1, n_2, n_3-1}^{2n+1-h+k} \left(u_1^{2n+1-h+k} u_1^{h-k} \right)_{1 < 2 < 3}$ times its multiplying factor has degree less than $3n+h+1$ in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , and less than h in n_3 . For $k_3=\ell_3=1$ by assumption III, the term has degree $3n+h+1$ in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , and h in n_3 . For $k_3 < \ell_3$ and $k_3 + \ell_3 \geq 3$

by assumption I $E_{n_1, n_2, n_3-1} \left(u_1^{2n+1-h+k_3} u_1^{h-l_3} \right)$ has degree less than $3n+h+1$ in all variables, $2n+1$ in n_3 , $2n+1$ in n_2 , and less than h in n_3 .

For $k_3 = 0$, $l_3 = 1$ and $k_3 = 0$, $l_3 = 2$ the corresponding terms have degree $3n+h+2$ ($3n+h+1$ if $k_3 = 0$, $l_3 = 2$) in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , $h(h-1$ if $k_3 = 0$, $l_3 = 2$) in n_3 .

The terms of highest degree in A' and B' of (4.3) with $k_1 = k_2 = 1$ and $l_1 = l_2 = 0$ cancel as in Case I. Likewise, the terms of highest degree with $k_1 = k_2 = k_3 = 0$ and $l_1 = l_2 = l_3 = 1$ from A' , B' , and C' of (4.3) cancel.

Thus $\psi E_{n_1, n_2, n_3} \left(u_1^{2n+1-h} u_1^h \right)$ has degree $3n+h+1$ in all variables, $2n+1$ in n_1 , $2n+1$ in n_2 , and h in n_3 and by lemma 1 $E_{n_1, n_2, n_3} \left(u_1^{2n+1-h} u_1^h \right)$ does also. This completes the proof of the lemma.

In what follows we assume that $n_i = m_i N$ for fixed m_i . The weakest condition for $n_1, n_2, n_3 \rightarrow \infty$ in the asymptotic theory is not known, but if $N \rightarrow \infty$ then $n_i \rightarrow \infty$ at the same rate. This condition is certainly sufficient for practical problems.

Corollary. For $i+j$ odd $E_{n_1, n_2, n_3} \left(u_1^i u_1^j \right) / \sigma_{u_1 < 2}^i \sigma_{u_1 < 2 < 3}^j \rightarrow 0$ as $N \rightarrow \infty$.

Proof. $\sigma_{u_1 < 2}^i \sigma_{u_1 < 2 < 3}^j$ has degree $(3/2)i + (5/2)j$ in all variables. Q.E.D.

As in [2] define

$$\rho_{n_1, n_2, n_3}^{\alpha\beta} \equiv E_{n_1, n_2, n_3} \left(u_1^\alpha u_1^\beta \right) / \sigma_{u_1 < 2}^\alpha \sigma_{u_1 < 2 < 3}^\beta.$$

In particular let

$$\rho_{n_1, n_2, n_3} \equiv E_{n_1, n_2, n_3} \left(u_1 < 2 u_1 < 2 < 3 \right) / \sigma_{u_1 < 2} \sigma_{u_1 < 2 < 3}$$

$$= \frac{n_1 n_2 n_3 (n_2 + 1) / 24}{\sqrt{n_1 n_2 (n_1 + n_2 + 1) / 12} \sqrt{n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4) / 180}}$$

Use $\rho_{n_1, n_2, n_3} \rightarrow \rho$ to mean $n_1, n_2, n_3 \rightarrow \infty$ in such a way that $\rho_{n_1, n_2, n_3} \rightarrow \rho$. For $n_i = m_i N$, ρ will depend on m_1, m_2 , and m_3 .

We need the following moments of bivariate normal random variables

$u_1 = U_1 - EU_1$, $u_2 = U_2 - EU_2$ with correlation coefficient ρ . The standardized joint moments are

$$\rho^{2i, 2j} = \frac{(2i)!(2j)!}{2^{i+j}} \sum_{\alpha=0}^{\min(i, j)} \frac{(2\rho)^{2\alpha}}{(i-\alpha)!(j-\alpha)!(2\alpha)!}$$

(4.4)

$$\rho^{2i+1, 2j+1} = \frac{(2i+1)!(2j+1)!}{2^{i+j}} \rho \sum_{\alpha=0}^{\min(i, j)} \frac{(2\rho)^{2\alpha}}{(i-\alpha)!(j-\alpha)!(2\alpha+1)!}$$

Theorem 4. The variables

$$\frac{U_1 < 2 - n_1 n_2 / 2}{\sqrt{n_1 n_2 (n_1 + n_2 + 1) / 12}}$$

and

$$\frac{U_1 < 2 < 3 - n_1 n_2 n_3 / 6}{\sqrt{n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4) / 180}}$$

are asymptotically jointly normal with means 0, variances 1, and correlation coefficient $\rho(m_1, m_2, m_3)$, where

$$\begin{aligned} \lim_{N \rightarrow \infty} \rho_{n_1, n_2, n_3} &= \frac{m_1 m_2 m_3 / 2}{\sqrt{(m_1^2 m_2 + m_1 m_2^2)(4m_1^2 m_2^2 m_3 + m_1^2 m_2 m_3^2 + 4m_1 m_2^2 m_3^2) / 15}} \\ &= \rho(m_1, m_2, m_3) \end{aligned}$$

Proof. Consider $i+j = 2$ first.

$$\lim_{N \rightarrow \infty} E_{n_1, n_2, n_3} \left(u_1^2 < 2 \right) / \sigma_{u_1 < 2}^2 = 1$$

$$\lim_{N \rightarrow \infty} E_{n_1, n_2, n_3} \left(u_1^2 < 2 < 3 \right) / \sigma_{u_1 < 2 < 3}^2 = 1$$

$$\lim_{N \rightarrow \infty} E_{n_1, n_2, n_3} \left(u_1 < 2 \quad u_1 < 2 < 3 \right) / \sigma_{u_1 < 2} \sigma_{u_1 < 2 < 3} = \rho$$

These satisfy (4.4). For $\alpha+\beta < i+j = 2n$ (say) we inductively assume $\rho_{n_1, n_2, n_3}^{\alpha\beta}$ satisfies (4.4) as $N \rightarrow \infty$. Call this assumption IV.

Now

$$\begin{aligned} &\psi E_{n_1, n_2, n_3} \left(u_1^{2n} < 2 \right) \\ &= \frac{2n(2n-1)}{2} n_1 \left(\frac{n_2}{2} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-2} < 2 \right) \\ &\quad + \frac{2n(2n-1)}{2} n_2 \left(\frac{n_1}{2} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-2} < 2 \right) \\ &\quad + P_{3n-1}(n_1, n_2, n_3) , \end{aligned}$$

where $P_{3n-1}(n_1, n_2, n_3)$ is a polynomial of degree $3n-1$ in all variables, at most $2n$ in n_1 , at most $2n$ in n_2 , and 0 in n_3 .

Combining terms and standardizing we get

$$\begin{aligned} & \psi_{E_{n_1, n_2, n_3}}^{(u_1^{2n} < 2)} / \sigma_{u_1 < 2}^{2n} \\ &= \frac{\frac{2n(2n-1)}{2} (1/4) n_1 n_2 (n_1 + n_2)}{(1/12) n_1 n_2 (n_1 + n_2 + 1)} E_{n_1, n_2, n_3}^{(u_1^{2n-2} < 2)} / \sigma_{u_1 < 2}^{2n-2} \\ &+ O(P'(N)) \end{aligned}$$

where $O(P'(N))$ means that $P'(N)$ tends to zero as $N \rightarrow \infty$.

$$\begin{aligned} & \lim_{N \rightarrow \infty} \psi_{E_{n_1, n_2, n_3}}^{(u_1^{2n} < 2)} / \sigma_{u_1 < 2}^{2n} \\ &= 3n[(2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1] \end{aligned}$$

by assumption IV. Then by lemma 2 $E_{n_1, n_2, n_3}^{(u_1^{2n} < 2)} / \sigma_{u_1 < 2}^{2n}$ satisfies (4.4)

For $j < r$ we inductively assume $\rho_{n_1, n_2, n_3}^{2n-j, j}$ satisfies (4.4) as $N \rightarrow \infty$. Call this assumption V. We need to consider two cases of j even and j odd.

Case I. $r = 2h$ (even)

$$\begin{aligned} & \psi_{E_{n_1, n_2, n_3}}^{(u_1^{2n-2h} < 2 \quad u_1^{2h} < 2 < 3)} \\ &= \frac{(2n-2h)(2n-2h-1)}{2} n_1 \left(\frac{n_2}{2}\right)^2 E_{n_1-1, n_2, n_3}^{(u_1^{2n-2h-2} < 2 \quad u_1^{2h} < 2 < 3)} \\ &+ (2n-2h)(2h) n_1 \left(\frac{n_2}{2}\right) \left(\frac{n_2 n_3}{6}\right) E_{n_1-1, n_2, n_3}^{(u_1^{2n-2h-1} < 2 \quad u_1^{2h-1} < 2 < 3)} \end{aligned}$$

$$\begin{aligned}
& + \frac{(2h)(2h-1)}{2} n_1 \left(\frac{n_2 n_3}{6} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-2h} u_1^{2h-2} \right) \\
& + \frac{(2n-2h)(2n-2h-1)}{2} n_2 \left(\frac{n_1}{2} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h-2} u_1^{2h} \right) \\
& + (2n-2h)(2h)(-1)n_2 \left(\frac{n_1}{2} \right) \left(\frac{n_1 n_3}{6} \right) E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h-1} u_1^{2h-1} \right) \\
& + \frac{(2h)(2h-1)}{2} n_2 \left(\frac{n_1 n_3}{6} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h} u_1^{2h-2} \right) \\
& + (2h) n_3 E_{n_1, n_2, n_3-1} \left(u_1^{2n-2h+1} u_1^{2h-1} \right) \\
& + \frac{(2h)(2h-1)}{2} n_3 \left(\frac{n_1 n_2}{3} \right)^2 E_{n_1, n_2, n_3-1} \left(u_1^{2n-2h} u_1^{2h-2} \right) \\
& + P_{3n+2h-1}(n_1, n_2, n_3)
\end{aligned}$$

where $P_{3n-2h-1}(n_1, n_2, n_3)$ is a polynomial of degree $3n-2h-1$ in all variables, at most $2n$ in n_1 , $2n$ in n_2 , and $2h$ in n_3 .

Since only the highest order terms are important, we can combine the above terms as follows:

$$\begin{aligned}
& \downarrow E_{n_1, n_2, n_3} \left(u_1^{2n-2h} u_1^{2h} \right) \\
& = \frac{(2n-2h)(2n-2h-1)}{2} (1/4) n_1 n_2 (n_1 + n_2) E_{n_1, n_2, n_3} \left(u_1^{2n-2h-2} u_1^{2h} \right) \\
& + (2n-2h)(2h)(1/12) n_1 n_2 n_3 (n_2 - n_1) E_{n_1, n_2, n_3} \left(u_1^{2n-2h-1} u_1^{2h-1} \right)
\end{aligned}$$

$$\begin{aligned}
 & + \frac{(2h)(2h-1)}{2} (1/36) n_1 n_2 n_3 (n_2 n_3 + n_1 n_3 + 4n_1 n_2) E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h} u_{1 < 2 < 3}^{2h-2} \right) \\
 & + (2h) n_3 E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h+1} u_{1 < 2 < 3}^{2h-1} \right) \\
 & + P_{3n+2h-1}^* (n_1, n_2, n_3) .
 \end{aligned}$$

Dividing by $\sigma_{u_{1 < 2}}^{2n-2h} \sigma_{u_{1 < 2 < 3}}^{2h}$ we obtain

$$\begin{aligned}
 & \psi E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h} u_{1 < 2 < 3}^{2h} \right) / \sigma_{u_{1 < 2}}^{2n-2h} \sigma_{u_{1 < 2 < 3}}^{2h} \\
 & = \frac{\frac{(2n-2h)(2n-2h-1)}{2} (1/4) n_1 n_2 (n_1 + n_2)}{(1/12) n_1 n_2 (n_1 + n_2 + 1)} \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h-2} u_{1 < 2 < 3}^{2h} \right)}{\sigma_{u_{1 < 2}}^{2n-2h-2} \sigma_{u_{1 < 2 < 3}}^{2h}} \\
 & + \frac{(2n-2h)(2h)(1/12) n_1 n_2 n_3 (n_2 + 1 - n_2 \left(\frac{n_1}{n_3} \right) - 1)}{\sqrt{(1/12) n_1 n_2 (n_1 + n_2 + 1)} \sqrt{1/180) n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4)}} \\
 & \times \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h-1} u_{1 < 2 < 3}^{2h-1} \right)}{\sigma_{u_{1 < 2}}^{2n-2h-1} \sigma_{u_{1 < 2 < 3}}^{2h-1}} \\
 & + \frac{\frac{(2r)(2r-1)}{2} (1/36) n_1 n_2 n_3 (n_2 n_3 + n_1 n_3 + 4n_1 n_2)}{(1/180) n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4)} \\
 & \times \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h} u_{1 < 2 < 3}^{2h-2} \right)}{\sigma_{u_{1 < 2}}^{2n-2h} \sigma_{u_{1 < 2 < 3}}^{2h-2}}
 \end{aligned}$$

$$+ \frac{(2h)n_3 \sqrt{(1/12)n_1 n_2 (n_1 + n_2 + 1)}}{\sqrt{(1/180)n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4)}}$$

$$\times \frac{E_{n_1, n_2, n_3} \left(\begin{matrix} u_1^{2n-2h+1} & u_1^{2h-1} \\ u_1 < 2 & u_1 < 2 < 3 \end{matrix} \right)}{\sigma_{u_1 < 2}^{2n-2h+1} \sigma_{u_1 < 2 < 3}^{2h-1}}$$

$$+ O(P'(N)).$$

Using assumptions IV and V and assuming $h < n-h$

$$\lim_{N \rightarrow \infty} \downarrow E_{n_1, n_2, n_3} \left(\begin{matrix} u_1^{2n-2h} & u_1^{2h} \\ u_1 < 2 & u_1 < 2 < 3 \end{matrix} \right) / \sigma_{u_1 < 2}^{2n-2h} \sigma_{u_1 < 2 < 3}^{2h}$$

$$\begin{aligned} &= 3 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(2\rho)^{2\alpha}}{(n-h-1-\alpha)!(h-\alpha)!(2\alpha)!} \\ &+ 8\rho \frac{(2n-2h)!(2h)!}{2^n} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-1-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\ &- 8\left(\frac{m_1}{m_2}\right)\rho \frac{(2n-2h)!(2h)!}{2^n} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-1-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\ &+ 5 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\ &- 15 \frac{m_1 m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3} \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \end{aligned}$$

$$+ K \frac{(2n-2h+1)!(2h)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!}$$

where K is either

$$(15/2) \frac{m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3} (1/\rho)$$

or
$$\left(2\left(\frac{m_1}{m_2}\right) + 2\right)\rho$$

depending on which way the limit is taken. Reducing further we get

$$\begin{aligned} & \lim_{N \rightarrow \infty} \psi E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h} u_{1 < 2 < 3}^{2h} \right) / \sigma_{u_{1 < 2}}^{2n-2h} \sigma_{u_{1 < 2 < 3}}^{2h} \\ &= 3n \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\ &+ 3 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(-h-\alpha)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\ &+ 2 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=1}^h \frac{(2\rho)^{2\alpha}(2\alpha)}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\ &- 2 \left(\frac{m_1}{m_2}\right) \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=1}^h \frac{(2\rho)^{2\alpha}(2\alpha)!}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\ &+ 5 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(h-\alpha)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \end{aligned}$$

$$\begin{aligned}
& - 15 \frac{m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3} \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(2p)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
& + 15 \frac{m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3} \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(2p)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
& + 2 \left(\frac{m_1}{m_2} \right) \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=1}^h \frac{(2p)^{2\alpha}(2\alpha)}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& + 2 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=1}^h \frac{(2p)^{2\alpha}(2\alpha)}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& = 3n \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(2p)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& + 3 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(-h-\alpha)(2p)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& + 5 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(h-\alpha)(2p)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& + 4 \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=1}^h \frac{(2p)^{2\alpha}(2\alpha)}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!} \\
& = (3n+2h) \frac{(2n-2h)!(2h)!}{2^n} \sum_{\alpha=0}^h \frac{(2p)^{2\alpha}}{(n-h-\alpha)!(h-\alpha)!(2\alpha)!}
\end{aligned}$$

By lemma 2 then $E_{n_1, n_2, n_3} \left(u_1^{2n-2h} u_2^{2h} \mid u_1 < 2, u_1 < 2 < 3 \right)$ satisfies (4.4) for $h < n - h$.

The same conclusion holds if $h \geq n - h$.

Case II. $r = 2h-1$ (odd).

$$\begin{aligned}
 & \psi E_{n_1, n_2, n_3} \left(u_1^{2n-2h+1} u_1^{2h-1} \right) \\
 &= \frac{(2n-2h+1)(2n-2h)}{2} n_1 \left(\frac{n_2}{2} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-2h-1} u_1^{2h-1} \right) \\
 &+ (2n-2h+1)(2h-1) n_1 \left(\frac{n_2}{2} \right) \left(\frac{n_2 n_3}{6} \right) E_{n_1-1, n_2, n_3} \left(u_1^{2n-2h} u_1^{2h-2} \right) \\
 &+ \frac{(2h-1)(2h-2)}{2} n_1 \left(\frac{n_2 n_3}{6} \right)^2 E_{n_1-1, n_2, n_3} \left(u_1^{2n-2h+1} u_1^{2h-3} \right) \\
 &+ \frac{(2n-2h+1)(2n-2h)}{2} n_2 \left(\frac{n_1}{2} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h-1} u_1^{2h-1} \right) \\
 &+ (2n-2h+1)(2h-1)(-1) n_2 \left(\frac{n_1}{2} \right) \left(\frac{n_1 n_3}{6} \right) E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h} u_1^{2h-2} \right) \\
 &+ \frac{(2h-1)(2h-2)}{2} n_2 \left(\frac{n_1 n_3}{6} \right)^2 E_{n_1, n_2-1, n_3} \left(u_1^{2n-2h+1} u_1^{2h-3} \right) \\
 &+ (2h-1) n_3 E_{n_1, n_2, n_3-1} \left(u_1^{2n-2h+2} u_1^{2h-2} \right) \\
 &+ \frac{(2h-1)(2h-2)}{2} n_3 \left(\frac{n_1 n_2}{3} \right)^2 E_{n_1, n_2, n_3-1} \left(u_1^{2n-2h+1} u_1^{2h-3} \right) \\
 &+ P_{3n-3h-3}(n_1, n_2, n_3)
 \end{aligned}$$

where $P_{3n-2h-2}(n_1, n_2, n_3)$ is a polynomial of degree $3n-2h-2$ in all variables, at most $2n$ in n_1 , $2n$ in n_2 and $2h-1$ in n_3 .

By combining and dividing by $\sigma_{u_1 < 2}^{2n-2h+1} \sigma_{u_1 < 2 < 3}^{2h-1}$ we obtain

$$\begin{aligned}
 & \psi_{n_1, n_2, n_3}^{E_{n_1, n_2, n_3}} \left(u_{1 < 2}^{2n-2h+1} u_{1 < 2 < 3}^{2h-1} \right) / \sigma_{u_{1 < 2}}^{2n-2h+1} \sigma_{u_{1 < 2 < 3}}^{2h-1} \\
 &= \frac{(2n-2h+1)(2n-2h)}{2} (1/4) n_1 n_2 (n_1 + n_2) \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h-1} u_{1 < 2 < 3}^{2h-1} \right)}{\sigma_{u_{1 < 2}}^{2n-2h-1} \sigma_{u_{1 < 2 < 3}}^{2h-1}} \\
 &+ \frac{(2n-2h+1)(2h-1)(1/12) n_1 n_2 n_3 (n_2 + 1 - n_2 \left(\frac{n_1}{n_3} \right) - 1)}{\sqrt{(1/12) n_1 n_2 (n_1 + n_2 + 1)} \sqrt{(1/180) n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4)}} \\
 &\times \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h} u_{1 < 2 < 3}^{2h-2} \right)}{\sigma_{u_{1 < 2}}^{2n-2h} \sigma_{u_{1 < 2 < 3}}^{2h-2}} \\
 &+ \frac{(2h-1)(2h-2)}{2} (1/36) n_1 n_2 n_3 (n_2 n_3 + n_1 n_3 + 4n_1 n_2) \\
 &\times \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h+1} u_{1 < 2 < 3}^{2h-3} \right)}{\sigma_{u_{1 < 2}}^{2n-2h+1} \sigma_{u_{1 < 2 < 3}}^{2h-3}} \\
 &+ \frac{(2h-1) n_3 \sqrt{(1/12) n_1 n_2 (n_1 + n_2 + 1)}}{\sqrt{(1/180) n_1 n_2 n_3 (4n_1 n_2 + n_1 n_3 + 4n_2 n_3 + 5n_1 + 2n_2 + 5n_3 + 4)}} \\
 &\times \frac{E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h+2} u_{1 < 2 < 3}^{2h-2} \right)}{\sigma_{u_{1 < 2}}^{2n-2h+2} \sigma_{u_{1 < 2 < 3}}^{2h-2}} \\
 &+ O(P'(N)) .
 \end{aligned}$$

Using assumptions IV and V and assuming $(h-1) < n-h$ we get

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} \downarrow E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h+1} u_{1 < 2 < 3}^{2h-1} \right) / \sigma_{u_{1 < 2}}^{2n-2h+1} \sigma_{u_{1 < 2 < 3}}^{2h-1} \\
 &= 3 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-1-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
 &+ 2\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
 &- 2\left(\frac{m_1}{m_2}\right)\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
 &+ 5 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-2} \frac{(2\rho)^{2\alpha}}{(n-h-2)!(h-2-\alpha)!(2\alpha+1)!} \\
 &- 15 \frac{\frac{m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3}}{(2n-2h+1)!(2h-1)!} \rho \sum_{\alpha=0}^{h-2} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-2-\alpha)!(2\alpha+1)!} \\
 &+ K \frac{(2n-2h+2)!(2h-1)!}{2^n} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h+1-\alpha)!(h-1-\alpha)!(2\alpha)!}
 \end{aligned}$$

where K is either

$$(15/2) \frac{\frac{m_2 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3}}{(1/\rho)}$$

or $\left(2\left(\frac{m_1}{m_2}\right) + 2\right)\rho$.

Reducing further we then have

$$\lim_{N \rightarrow \infty} \downarrow E_{n_1, n_2, n_3} \left(u_{1 < 2}^{2n-2h+1} u_{1 < 2 < 3}^{2h-1} \right) / \sigma_{u_{1 < 2}}^{2n-2h+1} \sigma_{u_{1 < 2 < 3}}^{2h-1}$$

$$\begin{aligned}
&= 3n \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&+ 3 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(-h-\alpha)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&+ 2\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
&- 2\left(\frac{m_1}{m_2}\right)\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
&+ 5 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-2} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-2-\alpha)!(2\alpha+1)!} \\
&- 15 \frac{\frac{m_1 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3}}{2^{n-1}} \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-2} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-2-\alpha)!(2\alpha+1)!} \\
&+ 15 \frac{\frac{m_1 m_3}{4m_1 m_2 + m_1 m_3 + 4m_2 m_3}}{2^{n-1}} \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=1}^{h-1} \frac{(2\alpha)(2\rho)^{2\alpha-2}}{(n-h+1-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
&+ 2\left(\frac{m_1}{m_2}\right)\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!} \\
&+ 2\rho \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha)!}
\end{aligned}$$

$$\begin{aligned}
&= 3n \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&+ 3 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(-h-\alpha)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&+ 5 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-2} \frac{(h-1-\alpha)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&+ 4 \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\alpha+1)(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!} \\
&= (3n+2h-1) \frac{(2n-2h+1)!(2h-1)!}{2^{n-1}} \rho \sum_{\alpha=0}^{h-1} \frac{(2\rho)^{2\alpha}}{(n-h-\alpha)!(h-1-\alpha)!(2\alpha+1)!}
\end{aligned}$$

By lemma 2 $E_{n_1, n_2, n_3} \left(u_1^{2n-2h+1} u_1^{2h-1} \right)$ satisfies (4.4) for $h-1 < n-h$. The same conclusion holds for $h-1 \geq n-h$. After appropriate standardization, the convergence of central moments to bivariate central normal moments implies a limiting bivariate normal distribution [6]. Thus the proof of the theorem is complete.

5. Conclusion

The asymptotic normality of $U_{1 < 2 < 3}$ permits a generalization over the two-sample Mann-Whitney statistic. Tables will be forthcoming in another report for small sample use of this statistic. These are easy to derive from the probability recursion equations in Section 2.

The development of a theory of nonparametric orthogonal comparisons will require a study of the properties of $U_{[1 < 2 < 3] \cup [1 < 2 < 3]}$. In addition, the properties of $U_{R\{ \}}$, where R represents more than two strict orderings, are not known.

Knowledge of error rates depends on the independence (perhaps asymptotic) under more general conditions than the null hypothesis. This subject is completely open.

References

- [1] H. B. Mann and D. R. Whitney, "On a Test of Whether One of Two Random Variables is Stochastically Larger Than the Other", Ann. Math. Statist., Vol. 18, 1947, pp. 50-60.
- [2] D. R. Whitney, "A Bivariate Extension of the U Statistic", Ann. Math. Statist., Vol. 22, 1951, pp. 274-282.
- [3] A. R. Jonckheere, "A Distribution-Free k-Sample Test Against Ordered Alternatives", Biometrika, Vol. 41, 1945, pp. 133-145.
- [4] M. L. Puri, "Some Distribution-Free k-Sample Rank Tests of Homogeneity Against Ordered Alternatives", Comm. on Pure and Applied Math., Vol. 18, 1965, pp. 51-63.
- [5] T. Beetle, Personal Communications.
- [6] D. A. S. Fraser, Nonparametric Methods in Statistics, John Wiley and Sons, 1957, pp. 208-209.